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The Invariant Imbedding Numerical Method for Fredholm Integral Equations with Degenerate Kernels

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Consider the Fredholm integral equation

$$u(t,x) = f(t) + \int_0^x k(t,y)\,u(y,x)\,dy, \qquad 0 \leqslant t \leqslant x,$$

where the kernel k has the form

$$k(t,y) = \sum_{i=1}^{M} g_i(t)\,h_i(y), \qquad 0 \leqslant t, y.$$

A Cauchy problem equivalent to the original integral equation is derived and validated. A general Fortran program has been written, and numerical results are displayed. Emphasis is on the inhomogeneous problem, though some remarks about the eigenvalue problem are given.

I. INTRODUCTION

The theory of invariant imbedding can be applied to Fredholm integral equations with various types of kernels, [1]–[5]. By regarding the solution at a fixed point as a function of the interval of integration, a differential equation is obtained; this equation, combined with knowledge of the solution for one interval length, enables us to produce the solution for other lengths. Such an initial-value problem is suitable for numerical computation.

In this paper we derive and validate an initial-value method for an integral equation whose kernel is degenerate. Computational results are given. The Fortran program is available from the authors.

Consider the Fredholm integral equation [6], [7]

$$u(t,x) = f(t) + \int_0^x k(t,y)\,u(y,x)\,dy, \qquad 0 \leqslant t \leqslant x, \tag{1}$$

where the kernel k has the form

$$k(t,y) = \sum_{i=1}^{M} g_i(t)\,h_i(y), \qquad 0 \leqslant t, y. \tag{2}$$

In the customary manner it is seen that

$$u(t,x) = f(t) + \int_0^x \sum_{i=1}^{M} g_i(t)\,h_i(y)\,u(y,x)\,dy. \tag{3}$$

By introducing the M new functions of x, $c_1(x), \ldots, c_M(x)$, according to the relationships

$$c_i(x) = \int_0^x h_i(y)\,u(y,x)\,dy, \qquad i = 1, 2, \ldots, M, \tag{4}$$

Eq. (3) becomes

$$u(t,x) = f(t) + \sum_{i=1}^{M} c_i(x) g_i(t). \tag{5}$$

By substituting from Eq. (5) into Eq. (4) we see that

$$c_i(x) = \int_0^x h_i(y) \left[f(y) + \sum_{m=1}^{M} c_m(x) g_m(y) \right] dy. \tag{6}$$

Define the functions a_i and b_{im} for $i, m = 1, 2, \ldots, M$ by means of the relations

$$a_i(x) = \int_0^x h_i(y) f(y) \, dy, \tag{7}$$

and

$$b_{im}(x) = \int_0^x h_i(y) g_m(y) \, dy. \tag{8}$$

Then Eq. (6) may be rewritten as the linear algebraic system

$$c_i(x) = a_i(x) + \sum_{m=1}^{M} b_{im}(x) c_m(x), \qquad 0 \leqslant x, i = 1, 2, \ldots, M. \tag{9}$$

II. Derivation

It is now expedient to adopt vector-matrix notation. Let a be the M-dimensional column vector whose ith component is $a_i(x)$. The M-dimensional column vector c has as its ith component $c_i(x)$, and B is the square matrix of order M whose ith row and jth column is $b_{ij}(x)$. Equations (9) can now be written

$$c = a + Bc. \tag{10}$$

Also introduce the M-dimensional resolvent matrix R whose ith row and jth column is $r_{ij}(x)$, in terms of which the solution of Eq. (10) is

$$c = a + Ra. \tag{11}$$

For x sufficiently small such a matrix certainly exists. It is seen that the matrices B and R are connected by the relation

$$R = B + BR. \tag{12}$$

Differentiate both sides of Eq. (10) with respect to x to obtain

$$c' = a' + B'c + Bc'. \tag{13}$$

Primes indicate differentiation with respect to x. In view of Eq. (11) it is seen that Eq. (13) can be solved for c' in the form

$$c' = (a' + B'c) + R(a' + B'c), \tag{14}$$

or

$$c' = (I + R)(a' + B'c). \tag{15}$$

This is an ordinary differential equation for the vector c; a' and B' are functions of the independent variable x. Next, a differential equation for the resolvent matrix R is obtained. Differentiation of both sides of Eq. (12) yields

$$R' = B' + B'R + BR'. \tag{16}$$

Again using Eq. (11) it is seen that

$$R' = (B' + B'R) + R(B' + B'R), \tag{17}$$

or

$$R' = (I + R)(B' + B'R). \tag{18}$$

This is the desired Riccati equation for the matrix R.

From their definitions it is known that a, B, and c fulfill the initial conditions

$$a(0) = 0, \tag{19}$$

$$B(0) = 0, \tag{20}$$

and

$$c(0) = 0. \tag{21}$$

Furthermore, for R we have

$$R(0) = 0. \tag{22}$$

The initial-value problem for the vector c and the matrix R is contained in Eqs. (15), (18), (21), and (22). It consists of $M^2 + M$ ordinary differential equations with known initial conditions. The solution of the Fredholm integral Eq. (1) is then provided by Eq. (5) for $0 \leqslant t \leqslant x$ and for $x \leqslant x_1$, where $0 \leqslant x \leqslant x_1$ is an interval on which the initial-value problem has a solution.

III. Validation

It is easy to show that if c is a solution of Eq. (10), then

$$f(t) + \sum_{m=1}^{M} c_m(x)\, g_m(t) = f(t) + \int_0^x k(t, y)\left[f(y) + \sum_{m=1}^{M} c_m(x)\, g_m(y)\right] dy; \tag{23}$$

and Eq. (5) provides a solution of the integral Eq. (1). It remains to demonstrate that the solution of the initial-value problem does indeed satisfy Eq. (10). First, introduce the M-dimensional matrix G by means of the relation

$$G = B + BR. \tag{24}$$

Differentiation shows that

$$\begin{aligned} G' &= B' + B'R + BR' \\ &= B' + B'R + B(I + R)(B' + B'R) \\ &= (I + B + BR)(B' + B'R) \\ &= (I + G)(B' + B'R). \end{aligned} \tag{25}$$

The initial condition on G is

$$G(0) = 0. \tag{26}$$

Standard theorems show that

$$G = R, \qquad 0 \leqslant x \leqslant x_1, \tag{27}$$

or

$$R = B + BR, \qquad 0 \leqslant x \leqslant x_1. \tag{28}$$

Then introduce the vector w to be

$$w = a + Bc. \tag{29}$$

Differentiation yields

$$\begin{aligned} w' &= a' + B'c + Bc' \\ &= a' + B'c + B(I + R)(a' + B'c) \\ &= (I + B + BR)(a' + B'c). \end{aligned} \tag{30}$$

According to Eq. (28), this becomes

$$w' = (I + R)(a' + B'c). \tag{31}$$

Since

$$w(0) = 0, \tag{32}$$

we have

$$w = c \tag{33}$$

and

$$c = a + Bc, \qquad 0 \leqslant x \leqslant x_1. \tag{34}$$

This completes the demonstration.

IV. Differential Equations for the Components of c and R

According to Eqs. (7), (8), and (15), we have

$$\begin{aligned} c_i' &= \sum_{j=1}^{M} (\delta_{ij} + r_{ij}) \left(h_j f(x) + \sum_{n=1}^{M} h_j g_n c_n \right) \\ &= h_i f(x) + \sum_{n=1}^{M} h_i g_n c_n + \sum_{j=1}^{M} r_{ij} \left(h_j f(x) + \sum_{n=1}^{M} h_j g_n c_n \right) \\ &= h_i \left\{ f(x) + \sum_n g_n c_n \right\} + f(x) \sum_j r_{ij} h_j + \sum_n g_n c_n \sum_j r_{ij} h_j. \end{aligned} \tag{35}$$

The final result is

$$c_i'(x) = \left\{ f(x) + \sum_{m=1}^{M} g_m(x) c_m(x) \right\} \left\{ h_i(x) + \sum_{m=1}^{M} r_{im}(x) h_m(x) \right\}, \tag{36}$$

with

$$c_i(0) = 0, \qquad i = 1, 2, \ldots, M. \tag{37}$$

For the function r_{ij} it follows from Eq. (18) that

$$r'_{ij}(x) = \sum_{m=1}^{M} (\delta_{im} + r_{im}) \left(h_m g_j + \sum_{n=1}^{M} h_m g_n r_{nj} \right)$$

$$= h_i g_j + \sum_{n=1}^{M} h_i g_n r_{nj} + \sum_{m=1}^{M} r_{im} h_m g_j + \sum_{n=1}^{M} g_n r_{nj} \sum_{m=1}^{M} r_{im} h_m$$

$$= h_i \left\{ g_j + \sum_{n=1}^{M} g_n r_{nj} \right\} + \left\{ g_j + \sum_{n=1}^{M} g_n r_{nj} \right\} \sum_{m=1}^{M} r_{im} h_m. \tag{38}$$

The differential equations for the components of the resolvent matrix R are

$$r'_{ij}(x) = \left\{ g_j(x) + \sum_{n=1}^{M} g_n(x) r_{nj}(x) \right\} \left\{ h_i(x) + \sum_{m=1}^{M} r_{im}(x) h_m(x) \right\},$$

$$i, j = 1, 2, \ldots, M. \tag{39}$$

The initial conditions are

$$r_{ij}(0) = 0, \qquad i, j = 1, 2, \ldots, M. \tag{40}$$

The solution of the Fredholm integral, Eq. (1), is then provided by Eq. (5) for $0 \leqslant x \leqslant x_1$.

The computational procedure is to numerically integrate the system of $M^2 + M$ differential Eqs. (36) and (39) with the initial conditions of Eqs. (37) and (40) until x attains the desired interval length. Then Eq. (5) is used to produce $u(t, x)$.

V. Computational Results

A Fortran program has been written to solve the initial-value problem described in the previous section and, hence, the integral equation Eq. (1). It employs subroutines written by J. Buell for an Adams–Moulton integration scheme with a Runge–Kutta start. The results of four numerical experiments are described in this section. A typical run on an IBM 7044 requires less than 30 seconds to execute.

First consider Eq. (1) with

$$k(t, y) = e^{-t} e^{-y}, \tag{41}$$

$$f(t) = 1. \tag{42}$$

The closed-form solution is

$$u(t, x) = 1 + c(x) e^{-t}, \qquad 0 \leqslant t \leqslant x, \tag{43}$$

where

$$c(x) = (2 - 2e^{-x})/(1 + e^{-2x}), \qquad 0 \leqslant x < \infty. \tag{44}$$

The results of the initial-value calculations using a step size of .0025 for $x = 1$ and $x = 2$, as well as the exact solution, are displayed in Table I.

TABLE I

NUMERICAL RESULTS FOR THE KERNEL e^{-t-y}

a. Interval length $x = 1.0$

t	$u(t,x)$ Initial-value method	$u(t,x)$ Exact
0	2.1135387	2.1135399
0.1	2.0075715	2.0075726
0.2	1.9116883	1.9116894
0.3	1.8249297	1.8249306
0.4	1.7464273	1.7464281
0.5	1.6753953	1.6753961
0.6	1.6111230	1.6111237
0.7	1.5529669	1.5529675
0.8	1.5003452	1.5003457
0.9	1.4527310	1.4527315
1.0	1.4096480	1.4096484

b. Interval length $x = 2.0$

t	$u(t,x)$ Initial-value method	$u(t,x)$ Exact
0	2.6982213	2.6982254
0.2	2.3903860	2.3903893
0.4	2.1383518	2.1383545
0.6	1.9320036	1.9320058
0.8	1.7630600	1.7630618
1.0	1.6247407	1.6247422
1.2	1.5114944	1.5114956
1.4	1.4187762	1.4187772
1.6	1.3428650	1.3428658
1.8	1.2807141	1.2807148
2.0	1.2298293	1.2298298

In the second example the kernel is

$$k(t, y) = e^t e^y, \tag{45}$$

and the forcing function is unchanged. This differs from the first example in that now $c(x)$ becomes infinite for a finite value of x. This value of x is the positive root of the equation

$$1 - \int_0^x e^{2y} \, dy = 0, \tag{46}$$

which is

$$x_{\text{crit}} = .54930615. \tag{47}$$

The calculation was performed for two interval lengths, $x = .95x_{\text{crit}}$ with step size of .00031 and $x = .99x_{\text{crit}}$ with step size of .00016. The exact solution is

$$u(t) = 1 + c(x)\,e^t, \qquad 0 \leqslant t \leqslant x, \tag{48}$$

$$c(x) = (2e^x - 2)/(3 - e^{2x}), \qquad 0 \leqslant x < x_{\text{crit}} \tag{49}$$

Results are shown in Table II.

TABLE II

NUMERICAL RESULTS FOR THE KERNEL e^{t+y}

a. Interval length $x = 0.52184$

t	$u(t,x)$ Initial-value method	$u(t,x)$ Exact
0	9.5563613	9.5549738
0.05218408	10.014723	10.013261
0.10436817	10.497638	10.496098
0.15655225	11.006424	11.004801
0.20873634	11.542465	11.540755
0.26092042	12.107221	12.105420
0.31310450	12.702231	12.700334
0.36528859	13.329116	13.327116
0.41747268	13.989582	13.987476
0.46965677	14.685430	14.683211
0.52184084	15.418554	15.416216

b. Interval length $x = 0.54381$

t	$u(t,x)$ Initial-value method	$u(t,x)$ Exact
0	44.578135	44.583903
0.05438131	47.013592	47.019683
0.10876262	49.585161	49.591592
0.16314393	52.300446	52.307237
0.21752523	55.167482	55.174652
0.27190654	58.194748	58.202318
0.32628785	61.391199	61.399193
0.38066917	64.766291	64.774731
0.43505047	68.330006	68.338919
0.48943177	72.092887	72.102299
0.54381308	76.066066	76.076003

In the third example the integral equation is

$$u(t) = e^t - t + \int_0^1 t(1 - e^{ty})\, u(y)\, dy, \qquad 0 \leqslant t \leqslant 1, \tag{50}$$

an equation with a nondegenerate kernel. This is to be solved by approximating the kernel by the polynomial

$$K_M(t, y) = -t\left[ty + \frac{(ty)^2}{2!} + \ldots + \frac{(ty)^M}{M!}\right], \tag{51}$$

for $M = 2$, 3, and 4. The exact solution is

$$u(t) = 1, \qquad 0 \leqslant t \leqslant 1;$$

and the results of the computations using the initial-value method with step size of .005 are displayed in Table III. They show that the approximation improves as M increases from 2 to 4, but the results for $M = 2$ are quite accurate in themselves.

TABLE III

RESULTS FOR THE THIRD EXAMPLE

t	$u(t,x)$ $M = 2$	$u(t,x)$ $M = 3$	$u(t,x)$ $M = 4$
0	1.0000000	1.0000000	1.0000000
0.1	.99993922	.99998941	.99999848
0.2	.99979839	.99995811	.99999374
0.3	.99972437	.99991608	.99998606
0.4	.99998641	.99989542	.99997808
0.5	1.0009889	.99996302	.99997751
0.6	1.0032858	1.0002349	1.0000014
0.7	1.0075960	1.0008919	1.0000820
0.8	1.0148211	1.0021971	1.0002739
0.9	1.0260644	1.0045148	1.0006632
1.0	1.0426519	1.0083320	1.0013792

The fourth example shows that eigenvalues or critical lengths may be obtained. The kernel is considered to be

$$k(t, y) = \lambda e^{t+y}, \tag{52}$$

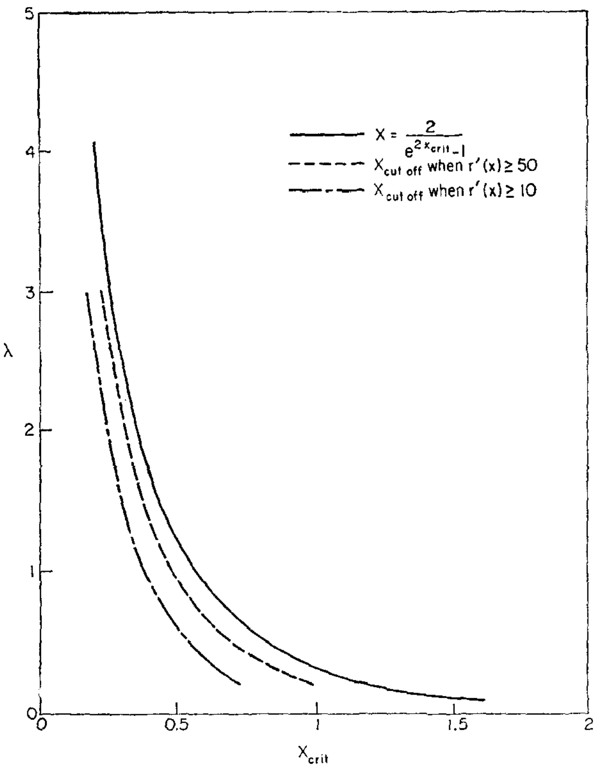
and

$$f = 0. \tag{53}$$

Then for various values of λ the differential equations for r and c are integrated with step size .005 until r' becomes sufficiently large. The exact relation is

$$\lambda = \frac{2}{e^{2x_{\text{crit}}} - 1}. \tag{54}$$

The results are shown graphically in the figure. Nonlinear extrapolation techniques are capable of producing extremely accurate results. Details will be given subsequently.



The first eigenvalue as a function of interval length

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